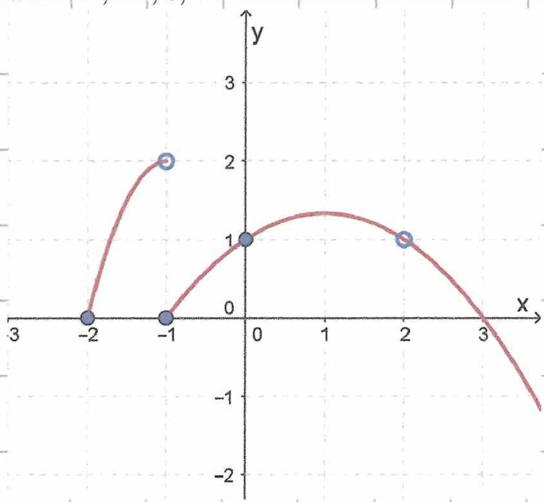
Limites : Solutions

1. Déterminer graphiquement les limites en a suivantes. Le cas échéant, déterminer la limite à gauche et la limite à droite.

   (a) $a = -2, -1, 0, 2$



$\lim\limits_{-2^-} f(x)\ \nexists$

$\lim\limits_{-2^+} f(x) = 0$

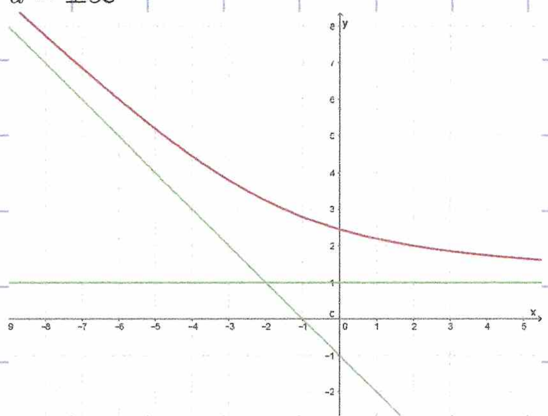
$\lim\limits_{-1^-} f(x) = 2$

$\lim\limits_{-1^+} f(x) = 0$

$\lim\limits_{0^-} f(x) = \lim\limits_{0^+} f(x) = 1$

$\lim\limits_{2^-} f(x) = \lim\limits_{2^+} f(x) = 1$

   (b) $a = \pm\infty$



$\lim\limits_{-\infty} f(x) = +\infty$

$\lim\limits_{+\infty} f(x) = 1$

3. Calculer les limites suivantes

(a) $\lim_{x\to 1} \frac{x^3-1}{x-1}$ $= \frac{0}{0}$ FI

$= \lim_{1} \frac{\cancel{(x-1)}(x^2+x+1)}{\cancel{(x-1)}} = \lim_{1} (x^2+x+1) = 3$

(b) $\lim_{x\to 0} \frac{4x^3-2x^2+x}{3x^2+2x}$ $= \frac{0}{0}$ FI

$= \lim_{0} \frac{\cancel{x}(4x^2-2x+1)}{\cancel{x}(3x+2)} = \frac{1}{2}$

(c) $\lim_{x\to 2} \frac{x^2+3x-10}{3x^2-5x-2} = \frac{0}{0}$ FI

$$\overset{\Delta}{=} \lim_{2} \frac{\cancel{(x-2)}(x+5)}{\cancel{(x-2)}(3x+1)} = 1$$

(d) $\lim_{x\to -1} \frac{x^3-x^2-x+1}{x^2+2x+1} = \frac{0}{0}$ FI

$$= \lim_{-1} \frac{(x^3-x^2)-(x-1)}{(x+1)^2} = \lim_{-1} \frac{x^2(x-1)-(x-1)}{D}$$

$$= \lim_{-1} \frac{(x-1)(x^2-1)}{D} = \lim_{-1} \frac{(x-1)^2\cancel{(x+1)}}{(x+1)^{\cancel{2}}}$$

$$= \pm\infty$$

(e) $\lim_{x\to-1} \frac{|x+1|}{x+1} = \frac{0}{0}$ FI

$$\lim_{-1^-} \frac{-(x+1)}{x+1} = -1$$

$$\lim_{-1^+} \frac{x+1}{x+1} = 1$$

car $|x+1| = \begin{cases} x+1 & \text{si } x \geq -1 \\ -(x+1) \end{cases}$

(f) $\lim_{x\to 0} \frac{\sqrt{1+x}-1}{x} = \frac{0}{0}$ FI

$$= \lim_{0} \frac{\sqrt{1+x}-1}{x} \cdot \frac{\sqrt{1+x}+1}{\sqrt{1+x}+1}$$

$$= \lim_{0} \frac{\cancel{1}+\cancel{x}-\cancel{1}}{\cancel{x}} \cdot \frac{1}{\sqrt{1+x}+1}$$

$$= \frac{1}{2}$$

(g) $\lim_{x\to 0} \frac{\sqrt{x^2+x+1}-1}{x} = \frac{0}{0}$ FI

$= \lim_{0} \frac{\sqrt{x^2+x+1}-1}{x} \cdot \frac{\sqrt{x^2+x+1}+1}{\sqrt{x^2+x+1}+1}$

$= \lim_{0} \frac{x^{\cancel{2}\,1}+\cancel{x}+\cancel{1}-\cancel{1}}{\cancel{x}} \cdot \frac{1}{\sqrt{x^2+x+1}+1}$

$N = x^2+x$

$= x(x+1)$

$= \frac{1}{2}$

(h) $\lim_{x\to 4} \frac{\sqrt{2x+1}-3}{\sqrt{x-2}-\sqrt{2}} = \frac{0}{0}$ FI

$= \lim_{4} \frac{\sqrt{2x+1}-3}{\sqrt{x-2}-\sqrt{2}} \cdot \frac{\sqrt{2x+1}+3}{\sqrt{2x+1}+3} \cdot \frac{\sqrt{x-2}+\sqrt{2}}{\sqrt{x-2}+\sqrt{2}}$

$= \lim_{4} \frac{2x+1-9}{x-2-2} \cdot \frac{\sqrt{x-2}+\sqrt{2}}{\sqrt{2x+1}+3}$

$= \lim_{4} \frac{2\,\cancel{x-8}^{\,2}}{\cancel{x-4}} \cdot \frac{\sqrt{x-2}+\sqrt{2}}{\sqrt{2x+1}+3}$

$= \frac{2 \cdot 2\sqrt{2}}{6}$

$= \frac{2\sqrt{2}}{3}$

4. Calculer les limites suivantes

(a) $\lim_{x \to \pm\infty} \frac{4x^3 - 2x^2 + 1}{x^2 - 1} = \frac{\infty}{\infty}$ FI

$= \lim_{\pm\infty} \frac{4x^{\cancel{3}}}{\cancel{x^2}} = \pm\infty$

(b) $\lim_{x \to \pm\infty} \frac{4x^3 - 2x^2 + 1}{3x^3 + 5} = \frac{\infty}{\infty}$ FI

$= \lim_{\pm\infty} \frac{4\cancel{x^3}}{3\cancel{x^3}} = \frac{4}{3}$

(c) $\lim\limits_{x\to\pm\infty} \dfrac{4x^3 - 2x^2 + 1}{x^4 - 1} = \dfrac{\infty}{\infty}$ FI

$$= \lim_{\pm\infty} \frac{4\cancel{x^3}}{x^{\cancel{4}}} = 0$$

(d) $\lim\limits_{x\to\pm\infty} \dfrac{4x + 1}{\sqrt{x^2 + 5x + 3}} = \dfrac{\infty}{\infty}$ FI

$$= \lim_{\pm\infty} \frac{4x}{\sqrt{x^2}} = \lim_{\pm\infty} \frac{4x}{|x|}$$

$$= \begin{cases} \lim\limits_{-\infty} \dfrac{4\cancel{x}}{-\cancel{x}} = -4 \\ \lim\limits_{+\infty} \dfrac{4\cancel{x}}{\cancel{x}} = 4 \end{cases}$$

(e) $\lim\limits_{x\to\pm\infty} \dfrac{2x}{\sqrt{2x^2-2x}} = \dfrac{\infty}{\infty}$ FI

$$= \lim_{\pm\infty} \frac{2x}{\sqrt{2x^2}} = \lim_{\pm\infty} \frac{2x}{\sqrt{2}\,|x|}$$

$$\Rightarrow \begin{cases} \lim\limits_{-\infty} \dfrac{2x}{-\sqrt{2}\,x} = -\sqrt{2} \\[2ex] \lim\limits_{+\infty} \dfrac{2x}{\sqrt{2}\,x} = \sqrt{2} \end{cases}$$

(f) $\cancel{\lim\limits_{x\to\pm\infty} \dfrac{x+1}{\sqrt{x^2-4}-x}}$

5. Calculer les limites suivantes

(a) $\lim_{x\to\pm\infty}\left[\sqrt{x^2+1}-x+1\right]$

$\lim_{-\infty}[\quad] = \infty + \infty = \infty$

$\lim_{+\infty}[\quad] = \infty - \infty$ FI

$= \lim_{+\infty}\left[\sqrt{x^2+1}-(x-1)\right]\cdot\frac{\sqrt{x^2+1}+(x-1)}{\sqrt{x^2+1}+(x-1)}$

x^2-2x+1

$= \lim_{+\infty}\frac{x^2+1-(x-1)^2}{\sqrt{x^2+1}+(x-1)} = \lim_{+\infty}\frac{2x}{D} = \frac{\infty}{\infty}$ FI

$= \lim_{+\infty}\frac{2x}{\sqrt{x^2}+x} = \lim_{+\infty}\frac{2x}{x+x} = 1$

(b) $\lim_{x\to\pm\infty}\left[x\left(\sqrt{x^2+1}-x\right)\right]$

$\lim_{-\infty}[\quad] = \infty + \infty = \infty$

$\lim_{+\infty}[\quad] = \infty - \infty$ FI

$= \lim_{+\infty}[\quad]\cdot\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x} = \lim_{+\infty}\frac{x(x^2+1-x^2)}{D}$

$= \lim_{+\infty}\frac{x}{\sqrt{x^2+1}+x} = \frac{\infty}{\infty}$ FI

$= \lim_{+\infty}\frac{x}{\sqrt{x^2}+x} = \lim_{+\infty}\frac{x}{x+x} = \frac{1}{2}$

(c) $\lim_{x\to\pm\infty}\left[\sqrt{x^2-x}-x\right]$

$$\lim_{-\infty}[\quad] = \infty + \infty = \infty$$

$$\lim_{+\infty}[\quad] = \infty - \infty \quad FI$$

$$= \lim_{+\infty}[\quad]\cdot\frac{\sqrt{x^2-x}+x}{\sqrt{x^2-x}+x} = \lim_{+\infty}\frac{\cancel{x^2}-x-\cancel{x^2}}{\sqrt{x^2+x}+x} = \frac{\infty}{\infty}\ FI$$

$$= \lim_{+\infty}\frac{-x}{\sqrt{x^2}+x} = \lim_{+\infty}\frac{-x}{x+x} = -\frac{1}{2}$$

(d) $\lim_{x\to\pm\infty}\left[3x+1-\sqrt{9x^2-3x}\right]$

$$\lim_{-\infty}[\quad] = -\infty - \infty = -\infty$$

$$\lim_{+\infty}[\quad] = \infty - \infty \quad FI$$

$$= \lim_{+\infty}[\quad]\cdot\frac{(3x+1)+\sqrt{9x^2-3x}}{(3x+1)+\sqrt{9x^2-3x}}$$

$9x^2+6x+1$

$$= \lim_{+\infty}\frac{(3x+1)^2-(9x^2-3x)}{3x+1+\sqrt{9x^2-3x}} = \lim_{+\infty}\frac{9x+1}{D} = \frac{\infty}{\infty}\ FI$$

$$= \lim_{+\infty}\frac{9x}{3x+\sqrt{9x^2}} = \lim_{+\infty}\frac{9x}{3x+3x} = \frac{9}{6} = \frac{3}{2}$$

6. Calculer les limites suivantes (cas mélangés)

(a) $\lim_{x\to\pm\infty} \frac{8x^3+2x^2-5x+1}{8x^3+10x^2-11x+2}$ $= \frac{\infty}{\infty}$ FI

$= \lim_{\pm\infty} \frac{\cancel{8x^3}}{\cancel{8x^3}} = 1$

(b) $\lim_{x\to 0} \left[\frac{x+4}{x^2-4} - \frac{x-6}{x(x-2)}\right]$

$= \lim_{0} \frac{x(x+4)-(x-6)(x+2)}{x(x-2)(x+2)}$

$= \lim_{0} \frac{x^2+4x-(x^2-4x-12)}{D}$

$= \lim_{0} \frac{8x+12}{x(x-2)(x+2)}$ (*)

$= \frac{12}{0}$

$= \pm\infty$

(c) $\lim\limits_{x \to \pm\infty} \dfrac{2x^2 + bx - b^2}{x^2 + 2bx + b^2}, b \in \mathbb{R}_0^+ = \dfrac{\infty}{\infty}$ FI

$$= \lim_{\pm\infty} \frac{2\cancel{x^2}}{\cancel{x^2}} = 2$$

(d) $\lim\limits_{x \to 2} \left[\dfrac{x+4}{x^2 - 4} - \dfrac{x-6}{x(x-2)} \right] = \lim\limits_{2} \dfrac{8x + 12}{x(x-2)(x+2)}$ cf (*)(b)

$$= \frac{28}{0}$$

$$= \pm\infty$$

(e) $\lim_{x\to\pm\infty}\left[\frac{x+4}{x^2-4}-\frac{x-6}{x(x-2)}\right] = \lim_{\pm\infty} \frac{8x+12}{x(x-2)(x+2)}$ f ⊛ (b)

$= \frac{\infty}{\infty}$ FI

$= \lim_{\pm\infty} \frac{8\cancel{x}}{x^{\cancel{3}\,2}} = 0$

(f) $\lim_{x\to -b}\frac{2x^2+bx-b^2}{x^2+2bx+b^2}, b\in\mathbb{R}_0^+ = \frac{0}{0}$ FI (1)

$\Delta_N = b^2 + 8b^2 = 9b^2 \Rightarrow x_{1,2} = \frac{-b\pm 3b}{4} < \begin{matrix}\frac{b}{2}\\ -b\end{matrix}$

$\Delta_D = 0 \Rightarrow x_{1,2} = -b$

(1) $= \lim_{-b} \frac{(2x-b)\cancel{(x+b)}}{(x+b)^{\cancel{2}}} = \frac{-3b}{0} = \pm\infty$

(g) $\lim\limits_{x\to\frac{1}{2}} \dfrac{8x^3+2x^2-5x+1}{8x^3+10x^2-11x+2} = \dfrac{0}{0}$ FI

(N)

	8	2	-5	1
$\frac{1}{2}$		4	3	-1
	8	6	-2	0

(D)

	8	10	-11	2
$\frac{1}{2}$		4	7	-2
	8	14	-4	0

$$\lim_{\frac{1}{2}} \frac{\cancel{\left(x-\frac{1}{2}\right)}\left(8x^2+6x-2\right)}{\cancel{\left(x-\frac{1}{2}\right)}\left(8x^2+14x-4\right)} = \frac{3}{5}$$

(h) $\lim\limits_{x\to 0} \dfrac{5+\dfrac{1}{x+1}}{x-\dfrac{x}{x-1}} = \lim\limits_{0} \dfrac{5x+5+1}{x+1} \quad \dfrac{x-1}{x^2-2x}$

$$= \lim_{0} \frac{(5x+6)(x-1)}{(x^2-2x)(x+1)} = -\frac{6}{0} = \pm\infty$$

(i) $\lim_{x\to\pm\infty} \dfrac{x+1}{\sqrt{x^2+2x+2}-1} = \dfrac{\infty}{\infty}$ FI

$= \lim_{\pm\infty} \dfrac{x}{\sqrt{x^2}} = \lim_{\pm\infty} \dfrac{x}{|x|}$

$\Rightarrow \begin{cases} \lim_{-\infty} \dfrac{x}{-x} = -1 \\ \lim_{+\infty} \dfrac{x}{x} = 1 \end{cases}$

(j) $\lim_{x\to 1} \dfrac{5+\dfrac{1}{x+1}}{x-\dfrac{x}{x-1}} = \lim_{1} \dfrac{(5x+6)(x-1)}{(x^2-2x)(x+1)}$ cf en (h)

$= 0$

(k) $\lim\limits_{x\to 0} \dfrac{\sqrt{x+4}-2}{x} = \dfrac{0}{0}$ FI

$= \lim\limits_{0} \dfrac{\sqrt{x+4}-2}{x} \cdot \dfrac{\sqrt{x+4}+2}{\sqrt{x+4}+2} = \lim\limits_{0} \dfrac{\cancel{x}+\cancel{4}-\cancel{4}}{\cancel{x}\left(\sqrt{x+4}+2\right)}$

$= \lim\limits_{0} \dfrac{1}{\sqrt{x+4}+2} = \dfrac{1}{4}$

(l) $\lim\limits_{x\to 2} \dfrac{5+\dfrac{1}{x+1}}{x-\dfrac{x}{x-1}} = \lim\limits_{2} \dfrac{(5x+6)(x-1)}{(x^2-2x)(x+1)} = \dfrac{16}{0} = \pm\infty$

(m) $\lim_{x\to\frac{1}{4}} \dfrac{8x^3+2x^2-5x+1}{8x^3+10x^2-11x+2} = \dfrac{0}{0}$ FI

N

	8	2	-5	1
$\frac{1}{4}$		2	1	-1
	8	4	-4	0

D

	8	10	-11	2
$\frac{1}{4}$		2	3	-2
	8	12	-8	0

$$= \lim_{\frac{1}{4}} \frac{\cancel{(x-\frac{1}{4})}(8x^2+4x-4)}{\cancel{(x-\frac{1}{4})}(8x^2+12x-8)} = \frac{-\frac{5}{2}}{-\frac{9}{2}} = \frac{5}{9}$$

(n) $\lim_{x\to\pm\infty} \dfrac{5+\dfrac{1}{x+1}}{x-\dfrac{x}{x-1}} = \lim_{\pm\infty} \dfrac{(5x+6)(x-1)}{(x^2-2x)(x+1)} = \dfrac{\infty}{\infty}$ FI

$$= \lim_{\pm\infty} \frac{5x \cdot x}{x^2 \cdot x} = \lim_{\pm\infty} \frac{5\cancel{x^2}}{x^{\cancel{3}}} = 0$$

(o) $\lim_{x\to\pm\infty} \frac{\sqrt{x+4}-2}{x} = \frac{\infty}{\infty}$ FI et $\lim_{-\infty} \nexists$ (cf dom f)

$= \lim_{+\infty} \frac{\sqrt{x}}{x} = \lim_{+\infty} \frac{\cancel{\sqrt{x}}}{(\sqrt{x})^{\cancel{2}}} = \lim_{+\infty} \frac{1}{\sqrt{x}} = 0$

(p) $\lim_{x\to 1} \frac{\sqrt{x}-1}{\sqrt[3]{x}-1} = \frac{0}{0}$ FI

$= \lim_{1} \frac{\sqrt{x}-1}{\sqrt[3]{x}-1} \cdot \frac{\sqrt{x}+1}{\sqrt{x}+1} \cdot \frac{(\sqrt[3]{x})^2+\sqrt[3]{x}+1}{(\sqrt[3]{x})^2+\sqrt[3]{x}+1}$

$= \lim_{1} \frac{\cancel{x-1}}{\cancel{x-1}} \cdot \frac{(\sqrt[3]{x})^2+\sqrt[3]{x}+1}{\sqrt{x}+1} = \frac{3}{2}$

(q) $\lim_{x\to -1} \frac{x+1}{\sqrt{x^2+2x+2}-1} = \frac{0}{0}$ FI

$$= \lim_{-1} \frac{x+1}{D} \cdot \frac{\sqrt{x^2+2x+2}+1}{\sqrt{x^2+2x+2}+1}$$

$$= \lim_{-1} \frac{\cancel{(x+1)}\left(\sqrt{x^2+2x+2}+1\right)}{\underbrace{x^2+2x+2-1}_{=x^2+2x+1=(x+1)^{\cancel{2}}}}$$

$$= \lim_{-1} \frac{\sqrt{x^2+2x+2}+1}{x+1} = \frac{2}{0} = \pm\infty$$

(r) $\lim_{x\to\pm\infty} \left[3x - \sqrt{9x^2-x-4}\right]$

$$\lim_{-\infty} [\ \ ] = -\infty - \infty = -\infty$$

$$\lim_{+\infty} [\ \ ] = \infty - \infty \quad \text{FI}$$

$$= \lim_{+\infty} [\ \ ] \cdot \frac{3x+\sqrt{9x^2-x-4}}{3x+\sqrt{9x^2-x-4}}$$

$$= \lim_{+\infty} \frac{9x^2-(9x^2-x-4)}{D} = \lim_{+\infty} \frac{x+4}{D} = \frac{\infty}{\infty} \quad \text{FI}$$

$$= \lim_{+\infty} \frac{x}{3x+\sqrt{9x^2}} = \lim_{+\infty} \frac{x}{3x+3x} = \frac{1}{6}$$

(s) $\lim\limits_{x\to 0} \dfrac{8x^3 + 2x^2 - 5x + 1}{8x^3 + 10x^2 - 11x + 2} = \dfrac{1}{2}$

(t) $\lim\limits_{x\to\pm\infty} \dfrac{\cancel{2x}}{\cancel{2x} - 3 + \sqrt{x^2 - 2x + 3}}$