Exercices complémentaires : Formules de transformation - Solutions

1. Calculer, en fonction des cosinus et des sinus de a, b et c, $\cos(b-c+a)$.

$$\cos(b-c+a) = \cos[(b-c)+a]$$
$$= \cos(b-c)\cos a - \sin(b-c)\sin a \quad (1)$$

En développant $\cos(b-c)$ et $\sin(b-c)$:

$$(1) = (\cos b\cos c + \sin b \sin c)\cos a - \ldots$$
$$\ldots (\sin b \cos c - \sin c \cos b)\sin a$$
$$= \cos a \cos b \cos c + \cos a \sin b \sin c - \ldots$$
$$\ldots \sin a \sin b \cos c + \sin a \cos b \sin c$$

2. Démontrer que $\dfrac{\sin(a+b)-\sin(a-b)}{\cos(a+b)-\cos(a-b)} = -\cot a$.

$$I = \frac{\cancel{\sin a \cos b} + \sin b \cos a - (\cancel{\sin a \cos b} - \sin b \cos a)}{\cancel{\cos a \cos b} - \sin a \sin b - (\cancel{\cos a \cos b} + \sin a \sin b)}$$

$$= \frac{\cancel{2} \cos a \cancel{\sin b}}{-\cancel{2} \sin a \cancel{\sin b}}$$

$$= -\cot a$$

$$= II$$

3. Calculer $\tan 3a$ en fonction de $\tan a$.

$$\tan 3a = \tan(2a + a)$$

$$= \frac{\tan 2a + \tan a}{1 - \tan 2a \tan a}$$

$$= \frac{\dfrac{2\tan a}{1-\tan^2 a} + \tan a}{1 - \dfrac{2\tan a}{1-\tan^2 a}\tan a}$$

$$= \frac{\dfrac{2\tan a + \tan a(1-\tan^2 a)}{\cancel{1-\tan^2 a}}}{\dfrac{1-\tan^2 a - 2\tan^2 a}{\cancel{1-\tan^2 a}}}$$

$$= \frac{3\tan a - \tan^3 a}{1 - 3\tan^2 a}$$

$$= \tan a \; \frac{3 - \tan^2 a}{1 - 3\tan^2 a}$$

4. Si $\tan x = 2$, calculer $\sin 4x$ et $\cos 4x$. $(x \in Q_I)$

. Si $\tan x = 2$, alors $\frac{1}{\cos^2 x} = 1 + 4$.

$\Leftrightarrow \cos^2 x = \frac{1}{5} \Leftrightarrow \cos x = \frac{\sqrt{5}}{5}$ $(\cos x \in Q_I)$

et $\sin x = \tan x . \cos x \Leftrightarrow \sin x = \frac{2\sqrt{5}}{5}$

. $\sin 4x = 2 \sin 2x \cos 2x$

$= 4 \sin x \cos x (2\cos^2 x - 1)$

$= 4 . \frac{2\sqrt{5}}{5} . \frac{\sqrt{5}}{5} . \left(2 . \frac{1}{5} - 1\right)$

$= \frac{40}{25} . -\frac{3}{5}$

$= \frac{-120}{125} = -\frac{24}{25}$

. $\cos 4x = 2\cos^2 2x - 1$

$= 2(2\cos^2 x - 1)^2 - 1$

$= 2 . \left(\frac{2}{5} - 1\right)^2 - 1$

$= 2 \frac{9}{25} - 1$

$= -\frac{7}{25}$

$\Rightarrow 4x \in Q_{III}$

5. Démontrer que $\frac{1}{\cot a - \cot 2a} = \sin 2a$.

$$I = \frac{1}{\frac{1}{\tan a} - \frac{1}{\tan 2a}} = \frac{1}{\frac{\tan 2a - \tan a}{\tan a \tan 2a}}$$

$$= \frac{\cancel{\tan a}\,\frac{2\tan a}{1-\tan^2 a}}{\frac{2\cancel{\tan a}}{1-\tan^2 a} - \cancel{\tan a}}$$

$$= \frac{\frac{2\tan a}{\cancel{1-\tan^2 a}}}{\frac{2-1+\tan^2 a}{\cancel{1-\tan^2 a}}}$$

$$= \frac{2\tan a}{1+\tan^2 a}$$

$$= 2\tan a \cos^2 a$$

$$= 2\,\frac{\sin a}{\cancel{\cos a}} \cdot \cos^{\cancel{2}} a$$

$$= 2\sin a \cos a$$

$$= \sin 2a$$

$$= II$$

6. Démontrer que $\dfrac{\tan\left(\frac{\pi}{4}+x\right)-\tan\left(\frac{\pi}{4}-x\right)}{\tan\left(\frac{\pi}{4}+x\right)+\tan\left(\frac{\pi}{4}-x\right)}=\sin 2x.$

$$I = \frac{\dfrac{\overbrace{\tan\frac{\pi}{4}}^{1}+\tan x}{1-\tan\frac{\pi}{4}\tan x}-\dfrac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}\tan x}}{\dfrac{1+\tan x}{1-\tan x}+\dfrac{1-\tan x}{1+\tan x}}$$

$$= \frac{\dfrac{(1+\tan x)^2-(1-\tan x)^2}{\cancel{1-\tan^2 x}}}{\dfrac{(1+\tan x)^2+(1-\tan x)^2}{\cancel{1-\tan^2 x}}}$$

$$= \frac{\cancel{1}+2\tan x+\cancel{\tan^2 x}-(\cancel{1}-2\tan x+\cancel{\tan^2 x})}{1+\cancel{2\tan x}+\tan^2 x+1-\cancel{2\tan x}+\tan^2 x}$$

$$= \frac{\cancel{4}^{2}\tan x}{\cancel{2}_{1}+\cancel{2}_{1}\tan^2 x}$$

$= \sin 2x$ (cf exercice 5)

$= II$

7. Démontrer que $\cos a + \cos 3a + \cos 5a + \cos 7a = 4 \cos 4a \cos 2a \cos a$.

$$
\begin{aligned}
\mathrm{I} &= (\cos a + \cos 3a) + (\cos 5a + \cos 7a) \\
&= 2 \cos 2a \cos(+a) + 2 \cos 6a \cos(+a) \\
&= 2 \cos a \left(\cos 2a + \cos 6a \right) \\
&= 2 \cos a \left(2 \cos 4a \cos(+2a) \right) \\
&= 4 \cos 4a \cos 2a \cos a \\
&= \mathrm{II}
\end{aligned}
$$

8. Démontrer que $\dfrac{\sin 2a + \cos 2a}{\cos a - \sin a - \cos 3a + \sin 3a} = \dfrac{1}{2\sin a}$.

Simplifions le dénominateur du 1^{er} membre (D_I)

$$D_I = (\cos a - \cos 3a) - (\sin a - \sin 3a)$$

$$= +2\sin(2a)\sin(+a) + [2\cos(2a)\sin(+a)]$$

$$= 2\sin 2a \sin a + 2\cos 2a \sin a$$

$$= 2\sin a(\sin 2a + \cos 2a)$$

$$I = \frac{\cancel{\sin 2a + \cos 2a}}{2\sin a(\cancel{\sin 2a + \cos 2a})}$$

$$= \frac{1}{2\sin 2a}$$

$$= II$$

9. Démontrer que $\sin 5a \sin a = \sin^2 3a - \sin^2 2a$.

$$\text{II} = (\sin 3a - \sin 2a)(\sin 3a + \sin 2a)$$

$$= 2 \cos \frac{5a}{2} \sin \frac{a}{2} \cdot 2 \sin \frac{5a}{2} \cos \frac{a}{2}$$

$$= \left(2 \sin \frac{5a}{2} \cos \frac{5a}{2}\right)\left(2 \sin \frac{a}{2} \cos \frac{a}{2}\right)$$

$$= \sin 5a \sin a$$

$$= \text{I}$$