Equations trigonométriques : Exercices complémentaires - Solutions

Résoudre les équations suivantes. Exprimer les solutions dans l'intervalle $]-\pi, \pi]$:

1. $2\sin 2x + 1 = 0$

$$\Leftrightarrow \sin 2x = -\frac{1}{2}$$

$$\Leftrightarrow \begin{cases} 2x = -\frac{\pi}{6} + 2k\pi \\ 2x = -\frac{5\pi}{6} + 2k\pi \end{cases}$$

$$\Leftrightarrow \begin{cases} x = -\frac{\pi}{12} + k\pi \\ x = -\frac{5\pi}{12} + k\pi \end{cases}$$

sur $]-\pi, \pi]$: $S : \left\{ -\frac{5\pi}{12}, -\frac{\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12} \right\}$

2. $2\cos 5x = -\sqrt{3}$

$$\cos 5x = -\frac{\sqrt{3}}{2} \Leftrightarrow \begin{cases} 5x = \frac{2\pi}{3} + 2k\pi \\ 5x = -\frac{2\pi}{3} + 2k\pi \end{cases}$$

$$\Leftrightarrow \begin{cases} x = \frac{2\pi}{15} + \frac{2k\pi}{5} \\ x = -\frac{2\pi}{15} + \frac{2k\pi}{5} \end{cases}$$

Sur $]-\pi, \pi]$:

$S : \left\{ -\frac{14\pi}{15}, -\frac{2\pi}{3}, -\frac{8\pi}{15}, -\frac{4\pi}{15}, -\frac{2\pi}{15}, \frac{2\pi}{15}, \frac{4\pi}{15}, \frac{8\pi}{15}, \frac{2\pi}{3}, \frac{14\pi}{15} \right\}$

3. $2\sin\left(\frac{\pi}{6}-2x\right)=\sqrt{3}$

$$\Leftrightarrow \sin\left(\frac{\pi}{6}-2x\right)=\frac{\sqrt{3}}{2} \Leftrightarrow \begin{cases} \frac{\pi}{6}-2x = \frac{\pi}{3}+2k\pi & (x) \\ \frac{\pi}{6}-2x = \frac{2\pi}{3}+2k\pi & (\pi-x) \end{cases}$$

$$\Leftrightarrow \begin{cases} -2x = \frac{\pi}{6}+2k\pi \\ -2x = \frac{\pi}{2}+2k\pi \end{cases} \Leftrightarrow \begin{cases} x = -\frac{\pi}{12}+k\pi \\ x = -\frac{\pi}{4}+k\pi \end{cases}$$

Sur $]-\pi,\pi]$: $S : \left\{-\frac{\pi}{4}, -\frac{\pi}{12}, \frac{3\pi}{4}, \frac{11\pi}{12}\right\}$

4. $3\tan\left(3x-\frac{\pi}{4}\right)=-\sqrt{3}$

CE $3x-\frac{\pi}{4} \neq \frac{\pi}{2}+k\pi$

$\Leftrightarrow 3x \neq \frac{3\pi}{4}+k\pi$

$\Leftrightarrow x \neq \frac{3\pi}{12}+\frac{k\pi}{3}$

$$\Leftrightarrow \tan\left(3x-\frac{\pi}{4}\right)=-\frac{\sqrt{3}}{3}$$

$$\Leftrightarrow 3x-\frac{\pi}{4}=-\frac{\pi}{6}+k\pi$$

$$\Leftrightarrow 3x=\frac{\pi}{12}+k\pi \Leftrightarrow x=\frac{\pi}{36}+\frac{k\pi}{3}$$

Sur $]-\pi,\pi]$:

$S : \left\{-\frac{35\pi}{36}, -\frac{23\pi}{36}, -\frac{11\pi}{36}, \frac{\pi}{36}, \frac{13\pi}{36}, \frac{25\pi}{36}\right\}$

5. $\sin 3x = \cos\left(\frac{\pi}{3} - x\right)$ $\Leftrightarrow \sin 3x = \sin\left[\frac{\pi}{2} - \left(\frac{\pi}{3} - x\right)\right]$

$\Leftrightarrow \sin 3x = \sin\left(\frac{\pi}{6} + x\right)$

$$\Leftrightarrow \begin{cases} 3x = \frac{\pi}{6} + x + 2k\pi & (\alpha) \\ 3x = \frac{5\pi}{6} - x + 2k\pi & (\pi - \alpha) \end{cases}$$

$$\Leftrightarrow \begin{cases} 2x = \frac{\pi}{6} + 2k\pi \\ 4x = \frac{5\pi}{6} + 2k\pi \end{cases} \Leftrightarrow \begin{cases} x = \frac{\pi}{12} + k\pi \\ x = \frac{5\pi}{24} + \frac{k\pi}{2} \end{cases}$$

Sur $]-\pi, \pi]$:

$S : \left\{ -\frac{11\pi}{12}, -\frac{19\pi}{24}, -\frac{7\pi}{24}, \frac{\pi}{12}, \frac{5\pi}{24}, \frac{17\pi}{24} \right\}$

6. $\sin 4x + \sin x = 0$

$\Leftrightarrow \sin 4x = -\sin x \Leftrightarrow \sin 4x = \sin(-x)$

$$\Leftrightarrow \begin{cases} 4x = -x + 2k\pi \\ 4x = \pi + x + 2k\pi \end{cases}$$

$$\Leftrightarrow \begin{cases} 5x = 2k\pi \\ 3x = \pi + 2k\pi \end{cases} \Leftrightarrow \begin{cases} x = \frac{2k\pi}{5} \\ x = \frac{\pi}{3} + \frac{2k\pi}{3} \end{cases}$$

Sur $]-\pi, \pi]$:

$S : \left\{ -\frac{4\pi}{5}, -\frac{2\pi}{5}, -\frac{\pi}{3}, 0, \frac{\pi}{3}, \frac{2\pi}{5}, \frac{4\pi}{5}, \pi \right\}$

7. $\sin\left(3x+\frac{\pi}{6}\right)-\cos\left(x-\frac{\pi}{3}\right)=0 \iff \sin\left(3x+\frac{\pi}{6}\right)=\cos\left(x-\frac{\pi}{3}\right)$

$\iff \sin\left(3x+\frac{\pi}{6}\right)=\sin\left[\frac{\pi}{2}-\left(x-\frac{\pi}{3}\right)\right]$

$\iff \sin\left(3x+\frac{\pi}{6}\right)=\sin\left(\frac{5\pi}{6}+x\right)$

$\iff \begin{cases} 3x+\frac{\pi}{6}=\frac{5\pi}{6}+x+2k\pi \\ 3x+\frac{\pi}{6}=\frac{\pi}{6}-x+2k\pi \end{cases}$ $\qquad \alpha \quad (\pi-\alpha)$

$\iff \begin{cases} 2x=\frac{2\pi}{3}+2k\pi \\ 4x=2k\pi \end{cases} \iff \begin{cases} x=\frac{\pi}{3}+k\pi \\ x=\frac{k\pi}{2} \end{cases}$

suite : voir verso ⊛

8. $2\cos^2 x-1=0 \iff \cos x=\pm\sqrt{\frac{1}{2}} \iff \cos x=\pm\frac{\sqrt{2}}{2}$

$\cos x=\frac{\sqrt{2}}{2}$

$\iff \begin{cases} x=\frac{\pi}{4}+2k\pi \\ x=-\frac{\pi}{4}+2k\pi \end{cases}$

$\cos x=-\frac{\sqrt{2}}{2}$

$\iff \begin{cases} x=\frac{3\pi}{4}+2k\pi \\ x=-\frac{3\pi}{4}+2k\pi \end{cases}$

Sur $]-\pi,\pi]$: $S=\left\{-\frac{3\pi}{4}, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}\right\}$

9. $\sin^2 x = \cos^2 x \iff \sin x = \pm \cos x$

$\sin x = \cos x$

$\iff \sin x = \sin\left(\frac{\pi}{2} - x\right)$

$\iff \begin{cases} x = \frac{\pi}{2} - x + 2k\pi & (\alpha) \\ x = \frac{\pi}{2} + x + 2k\pi & (\pi - \alpha) \end{cases}$

$\iff \begin{cases} 2x = \frac{\pi}{2} + 2k\pi \\ 0 = \frac{\pi}{2} + 2k\pi \Rightarrow \text{imp} \end{cases}$

$\iff x = \frac{\pi}{4} + k\pi$

$\sin x = -\cos x$

$\iff \sin x = \cos(\pi - x)$

$\iff \sin x = \sin\left[\frac{\pi}{2} - (\pi - x)\right]$

$\iff \sin x = \sin\left(x - \frac{\pi}{2}\right)$

$\iff \begin{cases} x = x - \frac{\pi}{2} + 2k\pi \\ x = \frac{3\pi}{2} - x + 2k\pi \end{cases}$

$\iff \begin{cases} 0 = -\frac{\pi}{2} + 2k\pi \Rightarrow \text{imp} \\ 2x = \frac{3\pi}{2} + 2k\pi \end{cases}$

$\iff x = \frac{3\pi}{4} + k\pi$

Sur $]-\pi, \pi]$: $S : \left\{ -\frac{3\pi}{4}, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4} \right\}$

10 $3\tan\theta = 2\cos\theta$

CE $\tan\theta \; \exists \Leftrightarrow \theta \neq \frac{\pi}{2} + k\pi$

$$3\,\frac{\sin\theta}{\cos\theta} = 2\cos\theta$$

$\Leftrightarrow 3\sin\theta - 2\cos^2\theta = 0$

$\Leftrightarrow 3\sin\theta - 2(1 - \sin^2\theta) = 0$

$\Leftrightarrow 2\sin^2\theta + 3\sin\theta - 2 = 0$

$$\Delta = 9 + 16 = 25 \Rightarrow \sin\theta_{1,2} = \frac{-3 \pm 5}{4} \left\langle \begin{array}{l} \frac{1}{2} \\ -2 \text{ AR.} \end{array} \right.$$

$$\sin\theta = \frac{1}{2} \Leftrightarrow \begin{cases} \theta = \frac{\pi}{6} + 2k\pi \\ \theta = \frac{5\pi}{6} + 2k\pi \end{cases}$$

Sur $]-\pi, \pi]$: $S = \left\{ \frac{\pi}{6}, \frac{5\pi}{6} \right\}$

1 $\sin\theta + \sin 2\theta + \sin 3\theta + \sin 4\theta = 0$

$$\Leftrightarrow \left(2\sin\frac{5\theta}{2}\cos\frac{3\theta}{2}\right) + \left(2\sin\frac{5\theta}{2}\cos\frac{\theta}{2}\right) = 0$$

$$\Leftrightarrow \cancel{2}\sin\frac{5\theta}{2}\left(\cos\frac{3\theta}{2} + \cos\frac{\theta}{2}\right) = 0$$

$$\Leftrightarrow \sin\frac{5\theta}{2}\ \cancel{2}\cos\theta\ \cos\frac{\theta}{2} = 0$$

$$\Leftrightarrow \begin{cases} \sin\frac{5\theta}{2} = 0 \Leftrightarrow \frac{5\theta}{2} = k\pi \Leftrightarrow \theta = \frac{2k\pi}{5} \\ \cos\theta = 0 \Leftrightarrow \theta = \frac{\pi}{2} + k\pi \\ \cos\frac{\theta}{2} = 0 \Leftrightarrow \frac{\theta}{2} = \frac{\pi}{2} + k\pi \Leftrightarrow \theta = \pi + 2k\pi \end{cases}$$

Sur $]-\pi, \pi]$: $S = \left\{-\frac{\pi}{2}, \frac{\pi}{2}, \pi\right\}$

CE : $\theta \neq \frac{\pi}{2} + k\pi, \theta \neq \frac{\pi}{8} + \frac{k\pi}{4}$

12 $\tan 4\theta = 4\tan\theta$ (1)

$$\tan 4\theta = \frac{2\tan 2\theta}{1-\tan^2 2\theta} = \frac{2\,\frac{2\tan\theta}{1-\tan^2\theta}}{1-\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)^2}$$

$$= \frac{\frac{4\tan\theta}{1-\tan^2\theta}}{\frac{1-2\tan^2\theta+\tan^4\theta-4\tan^2\theta}{(1-\tan^2\theta)^2}}$$

$$= \frac{4\tan\theta\,(1-\tan^2\theta)^{\cancel{2}}}{\cancel{(1-\tan^2\theta)}\,(\tan^4\theta-6\tan^2\theta+1)}$$

$$(1) \iff \frac{\cancel{4}\tan\theta\,(1-\tan^2\theta)}{\tan^4\theta-6\tan^2\theta+1} - \cancel{4}\tan\theta = 0$$

$$\iff \tan\theta\left[\frac{1-\tan^2\theta}{D} - 1\right] = 0$$

$$\iff \tan\theta\,(1-\tan^2\theta-\tan^4\theta+6\tan^2\theta-1) = 0$$

$$\iff \tan\theta\,(-\tan^4\theta+5\tan^2\theta) = 0$$

$$\iff \tan^3\theta\,(5-\tan^2\theta) = 0$$

$$\iff \begin{cases} \tan\theta = 0 \iff \theta = k\pi \\ 5-\tan^2\theta = 0 \iff \tan\theta = \pm\sqrt{5} \end{cases}$$

$$\iff \theta = \pm 1{,}15 + k\pi$$

Sur $]-\pi, \pi]$: $S: \{-1{,}99 ; -1{,}15 ; 1{,}15 ; 1{,}99\}$

13 $\cos x = \sin \frac{\pi}{8}$

$\cos x = \cos \frac{3\pi}{8}$ $\left(\frac{\pi}{2} - \frac{\pi}{8}\right)$

$\Leftrightarrow \begin{cases} x = \frac{3\pi}{8} + 2k\pi \\ x = -\frac{3\pi}{8} + 2k\pi \end{cases}$

Sur $]-\pi, \pi]$: $S : \left\{-\frac{3\pi}{8}, \frac{3\pi}{8}\right\}$

14. $\sin\left(3x+\frac{\pi}{3}\right) = \sin\left(\frac{2\pi}{3} - x\right)$

$$\Leftrightarrow \begin{cases} 3x + \frac{\pi}{3} = \frac{2\pi}{3} - x + 2k\pi \\ 3x + \frac{\pi}{3} = \pi - \left(\frac{2\pi}{3} - x\right) + 2k\pi \end{cases}$$

$$\Leftrightarrow \begin{cases} 4x = \frac{\pi}{3} + 2k\pi \\ 2x = 2k\pi \end{cases}$$

$$\Leftrightarrow \begin{cases} x = \frac{\pi}{12} + \frac{k\pi}{2} \\ x = k\pi \end{cases}$$

Sur $]-\pi, \pi]$: $S : \left\{-\frac{11\pi}{12}, -\frac{5\pi}{12}, 0, \frac{\pi}{12}, \frac{7\pi}{12}, \pi\right\}$

15. $2\sin^2 x + \sin x - 1 = 0$

$\Delta = 1 + 8 = 9 \qquad \sin x_{1,2} = \dfrac{-1 \pm 3}{4} \left\langle \begin{matrix} \frac{1}{2} \\ -1 \end{matrix} \right.$

$\sin x = \dfrac{1}{2} \Leftrightarrow \begin{cases} x = \dfrac{\pi}{6} + 2k\pi \\ x = \dfrac{5\pi}{6} + 2k\pi \end{cases}$

$\sin x = -1 \Leftrightarrow x = \dfrac{3\pi}{2} + 2k\pi = -\dfrac{\pi}{2} + 2k\pi$

S in $]-\pi, \pi]$: $S : \left\{ -\dfrac{\pi}{2}, \dfrac{\pi}{6}, \dfrac{5\pi}{6} \right\}$

16. $\sin 2x = \tan x$ CE : $x \neq \frac{\pi}{2} + k\pi$

$$2 \sin x \cos x = \frac{\sin x}{\cos x}$$

$$\Leftrightarrow 2 \sin x \cos^2 x - \sin x = 0$$

$$\Leftrightarrow \sin x \left(2 \cos^2 x - 1\right) = 0$$

$$\Leftrightarrow \begin{cases} \sin x = 0 & \Leftrightarrow x = k\pi \\ 2\cos^2 x - 1 = 0 & \Leftrightarrow \cos x = \pm \frac{\sqrt{2}}{2} \end{cases}$$

$$\Leftrightarrow \begin{cases} x = \pm \frac{\pi}{4} + 2k\pi \\ x = \pm \frac{3\pi}{4} + 2k\pi \end{cases}$$

$$S : \left\{0, \frac{\pi}{4}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{7\pi}{4}\right\} + 2k\pi$$

Sur $]-\pi, \pi]$: $\left\{-\frac{3\pi}{4}, -\frac{\pi}{4}, 0, \frac{\pi}{4}, \frac{3\pi}{4}, \pi\right\}$

17. $\cos 2x = \sin x$

$\Leftrightarrow 1 - 2\sin^2 x - \sin x = 0$

$\Leftrightarrow -2\sin^2 x - \sin x + 1 = 0$

$\Delta = 1 + 8 = 9 \quad \sin x_{1,2} = \frac{1 \pm 3}{-4} \begin{cases} -1 \\ \frac{1}{2} \end{cases}$

$\sin x = -1 \Leftrightarrow x = \frac{3\pi}{2} + 2k\pi = -\frac{\pi}{2} + 2k\pi$

$\sin x = \frac{1}{2} \Leftrightarrow \begin{cases} x = \frac{\pi}{6} + 2k\pi \\ x = \frac{5\pi}{6} + 2k\pi \end{cases}$

Sur $]-\pi, \pi]$: $S : \{ -\frac{\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6} \}$

18. $\dfrac{1+\sin x}{1-\sin x}=3$

CE : $1-\sin x \neq 0 \Leftrightarrow x \neq \frac{\pi}{2}+2k\pi$

$\Leftrightarrow 1+\sin x = 3-3\sin x$

$\Leftrightarrow 4\sin x - 2 = 0 \Leftrightarrow \sin x = \frac{1}{2}$

$$\Leftrightarrow \begin{cases} x = \frac{\pi}{6}+2k\pi \\ x = \frac{5\pi}{6}+2k\pi \end{cases}$$

Sur $]-\pi,\pi]$: $S : \left\{\frac{\pi}{6}, \frac{5\pi}{6}\right\}$

19. $\sin 2x \tan x - \tan x - \sin 2x + 1 = 0$ CE : $x \neq \frac{\pi}{2} + k\pi$

$\Leftrightarrow \tan x (\sin 2x - 1) - (\sin 2x - 1) = 0$

$\Leftrightarrow (\sin 2x - 1)(\tan x - 1) = 0$

$$\Leftrightarrow \begin{cases} \sin 2x = 1 \Leftrightarrow 2x = \frac{\pi}{2} + 2k\pi \\ \qquad\qquad \Leftrightarrow x = \frac{\pi}{4} + k\pi \\ \tan x = 1 \Leftrightarrow x = \frac{\pi}{4} + k\pi \end{cases}$$

Sur $]-\pi, \pi]$: $S = \left\{ -\frac{3\pi}{4}, \frac{\pi}{4} \right\}$

20) $\sin x + \cos x = \frac{\sqrt{2}}{2}$

On utilise les formules en $\tan \frac{x}{2}$ démontrées en cours

Si on pose $t = \tan \frac{x}{2}$

$$\Rightarrow \frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2} = \frac{\sqrt{2}}{2}$$

$$\Leftrightarrow 2t + 1 - t^2 - \frac{\sqrt{2}}{2}(1+t^2) = 0$$

$$\Leftrightarrow 4t + 2 - 2t^2 - \sqrt{2} - \sqrt{2}t^2 = 0$$

$$\Leftrightarrow -(2+\sqrt{2})t^2 + 4t + (2-\sqrt{2}) = 0$$

$$\Delta = 16 + 4(2+\sqrt{2})(2-\sqrt{2})$$

$$= 24$$

$$t_{1,2} = \frac{-4 \pm 2\sqrt{6}}{-2(2+\sqrt{2})} = -\sqrt{2} + 2 \pm (\sqrt{6} - \sqrt{3})$$

$\cdot\ t_1 = \ldots \Leftrightarrow \frac{x}{2} = -\frac{\pi}{24} + k\pi$

$\Leftrightarrow x = -\frac{\pi}{12} + 2k\pi$

$\cdot\ t_2 = \ldots \Leftrightarrow \frac{x}{2} = \frac{7\pi}{24} + k\pi$

$\Leftrightarrow x = \frac{7\pi}{12} + 2k\pi$

Sur $]-\pi, \pi]$: $S : \left\{ -\frac{\pi}{12}, \frac{7\pi}{12} \right\}$

21. $2\sin z + 3\cos z = 1$

De même qu'à l'exercice précédent :

$$\frac{4t}{1+t^2} + 3\,\frac{1-t^2}{1+t^2} = 1$$

$\iff 4t + 3 - 3t^2 - 1 - t^2 = 0$

$\iff -4t^2 + 4t + 2 = 0$

$\Delta = 16 + 32 = 48$

$$t_{1,2} = \frac{-4 \pm 4\sqrt{3}}{-8} = \frac{1 \pm \sqrt{3}}{2}$$

$t_1 = \frac{1+\sqrt{3}}{2} \iff \frac{x}{2} = 0{,}94 + k\pi$

$\iff x = 1{,}88 + 2k\pi$

$t_2 = \frac{1-\sqrt{3}}{2} \iff \frac{x}{2} = -0{,}35 + k\pi$

$\iff x = -1{,}7 + 2k\pi$

Sur $]-\pi, \pi]$: $S : \{-1{,}7 ; 1{,}88\}$